

On the assertion that PCT violation implies Lorentz non-invariance

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Abstract

Out of conviction or expediency, some current research programs [1–4] take for granted that “PCT violation implies violation of Lorentz invariance”. We point out that this claim [5] is still on somewhat shaky ground. In fact, for many years there has been no strengthening of the evidence in this direction. However, using causal perturbation theory, we prove here that when starting with a local PCT-invariant interaction, PCT symmetry can be maintained in the process of renormalization.

1 Introduction

In dealing with fundamental questions of science, it may be advisable to take a cue from knowledgeable philosophers. A recent account on PCT¹ invariance by one such [6] registers the fact that arguments for PCT conservation in relativistic field theory fall into two neatly separated classes. Heuristic treatments essentially amount to observing that “it does not appear possible to construct a

¹Since we stand from *causal perturbation theory*, we do not plump for the nowadays popular “CPT”.

‘reasonable’ interaction which violates PCT” [7]. Rigorous proofs were often based on the Wightman axioms² or the axioms of algebraic QFT —the exceedingly beautiful theory in [11] comes to mind— so holding within their limited domain of validity.

On this stark dichotomy we quote [6]. “We are thus faced with the following dilemma: (A) The Weinberg and textbook Lagrangian formalisms are complete but typically mathematically ill-formed. (B) The axiomatic and algebraic formalisms are incomplete but mathematically well-formed.” *Completeness* in context means that non-trivial, realistic interacting quantum field models can be formulated in the approach.

However, one paper in the literature presumes to bridge the chasm, claiming in its title “PCT violation implies violation of Lorentz invariance”. Beyond preambles, reaching the conclusion there takes a grand total of 20 half-lines and *one* displayed formula [5]. Such a *blitzkrieg* might breed distrust. Ponder for instance the classic proof by Epstein of PCT invariance of the S -matrix for theories of local observables [12]. It goes for eighteen tight pages of complex and functional analysis. Within its framework it remains state of the art —consult [13]. At its end Epstein declares “... it hardly needs to be remarked that the result is not expected to strengthen the evidence for the PCT invariance of nature”.

What about the literature on PCT conservation for *particular* interacting models in the realm of renormalized, perturbative QFT? On the basis of the Glaser–Lehmann–Zimmermann (GLZ) theorem [14], Steinmann undertook in [15] to construct a Wightman-like perturbative expansion for the ostensibly PCT-invariant self-interaction vertex of a neutral scalar field. After formidable prerequisites, his proof of PCT symmetry for the model goes on for more than ten pages. A different tack was taken in [16]: perturbative QED is developed in terms of time-ordered products (TOP), constructed by means of causal renormalization in the manner of Epstein and Glaser [17]. Painstakingly as well, Scharf there manages to show that TOP can be forged in a PCT invariant way. (His method is arguably valid for any model involving P-, C- and T-invariant vertices to begin with.)

Surely that was enough for many an expert not to bother with [5]. However, the stakes have become higher of late. For an incautious reader of [5], a failure of PCT conservation in nature would lead “beyond special relativity” automatically. Since vast current research programs on possible violation of Lorentz and PCT symmetries in nature appear to assume this, to keep ignoring the issue will not do. As well, experimental results as diverse as [18] and [19],³ where Lorentz invariance and symptoms of apparent PCT violation seem to coexist, must give us pause.

In the next section of this letter we try to puzzle out the argument of [5]. We point out that the assumptions needed to apply it have not been verified for any non-trivial, realistic model. In

²Among the treatments of Wightman theory, the traditional ones [8, 9] remain the best, in our mind. With some tweaking of the original axioms —exemplified in [10] for the Maxwell field— this method does prove PCT symmetry for many kinds of free fields.

³Long-standing tensions in the neutrino sector motivated the path-breaking paper [20]. Now the tension has been relieved by the update [21].

Section 3 we review the treatment of interacting fields in the framework of causal perturbation theory. This prepares the ground for Section 4, where we strengthen the perturbative treatment of PCT symmetry by a different path altogether. Section 5 contains our final situation assessment.

2 A bridge too far

An interacting QFT model is defined in [5] as Lorentz invariant if the “ τ -functions” (vacuum expectation values of TOP) are Lorentz covariant. We do not take exception to this. The τ -functions are exhibited there as

$$\tau^{(n)}(x_1, \dots, x_n) := \sum_P H(t_{P_1}, \dots, t_{P_n}) W^{(n)}(x_{P_1}, \dots, x_{P_n}); \quad (1)$$

where $x_i = (t_i, \mathbf{x}_i)$, the sum is over permutations P of n points, $W^{(n)}$ denotes putative Wightman functions, and H is the Heaviside function which enforces $t_{P_1} \geq \dots \geq t_{P_n}$. This is *the* formula. Then comes the punch line: the W -functions are to be shown to be *weakly local*, so PCT must hold.

Bold identities like (1) are haunted by the question of existence. In plain language: it is not quite clear what either side of the formula means. Helpfully, aside from vacuum expectation values having been taken, we notice that (1) is identical in form to (38) in [17]. There the fields being time-ordered are Wick polynomials of *free* (incoming) fields, and the formula is given as a *tentative definition* of its left hand side. As such it is nothing but the “solution” for TOP in terms of unrenormalized Feynman graphs. Epstein and Glaser hasten to indicate that expressions like the right hand side in (1) are illegitimate for $n > 2$, since the $W^{(n)}$ are distributions, whose product with the Heaviside functions is undefined: this is merely an instance of the ultraviolet problem of perturbation theory. On the face of it equation (1), as it appears in [5], ignores the need for renormalization at its peril.

Of course, the author of [5] is referring to W -functions for interacting fields. The issue of bad definition of (1) still stands. Please bear with us, as we attempt to salvage it. It may be argued that, whatever the messy avatars of a proper construction for the left hand side,⁴ it is to respect (1) insofar as $t_i \neq t_k$ for all $i < k$ (note that asking solely for $x_i \neq x_k$ for all $i < k$ would be insufficient). So begin with any point (x_1, \dots, x_n) such that the sum of linear combinations of the differences $x_i - x_{i-1}$ with non-negative coefficients (one at least being nonzero) is space-like, moreover fulfilling $t_1 > t_2 > \dots > t_n$; such points do exist. One may use (1) there. Then perform a Lorentz transformation such that the transformed point (x'_1, \dots, x'_n) satisfies $t'_n > \dots > t'_2 > t'_1$; such Lorentz transformations exist. One may again enforce (1) for (x'_1, \dots, x'_n) . Hence Lorentz invariance of the τ - and W -functions implies weak local commutativity for such points (x_1, \dots, x_n) :

$$W^{(n)}(x_1, \dots, x_n) = W^{(n)}(x_n, \dots, x_1).$$

⁴It may involve new couplings and even new fields [22].

This suffices for PCT invariance of the W -functions and PCT covariance of the fields, by a well-trodden argument by Jost [9], *provided* that Wightman’s axiom 0 (about the state space), axiom I (about the domain and continuity of the fields) and axiom II (about the Lorentz covariance of the fields) are verified—we borrow the numbering of the axioms from [8]. Namely, Jost’s proof crucially uses the analyticity properties of the W -functions, which follow from these axioms. The fact that Greenberg mentions this kind of additional assumptions only in a footnote to [5] may have caused misinterpretations of his statement.

Obviously the discussion in [5] may be relevant only for interacting models satisfying the Wightman axioms except *local commutativity*, that is to say

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0, \quad (2)$$

for the case of a boson field ϕ (this is Wightman’s axiom III). To wit, local commutativity implies trivially weak local commutativity, and hence for a model satisfying (2) one may dispense with the above contortions, since one immediately faces the question of applicability of Jost’s proof.⁵

If we were dealing with Wick polynomials of free fields and their W - and τ -functions, for which the Wightman axioms hold true, all we would obtain from the argument in [5] is PCT invariance of a *free* theory. Now, to the best of our knowledge, for non-trivial realistic models one cannot ascertain analyticity of Wightman-like functions; hence the argument *à la Jost* in [5] flounders. While the assertion that PCT conservation holds for everyday interacting relativistic theories remains plausible, to the question whether it has been proved at the required level of rigour, the clear and present answer is: only for a class of models—for instance QED in [16] as above said—and for none by Greenberg’s argument.

3 On interacting fields in causal perturbation theory

To deal seriously with interacting W -functions, we naturally have recourse to a rigorous theoretical framework. Causal perturbation theory by Epstein and Glaser “is closest to the spirit of Wightman’s axioms” [23] and well suited for our purpose. There are three steps. It entails first constructing the TOP of Wick polynomials; second, employing them to derive the interacting fields; third, performing the adiabatic limit (if available).

We sketch now the necessary detour. The Epstein–Glaser procedure was developed on the footsteps of Stückelberg [24], Bogoliubov [25] and Nishijima [26]. Let W be the vector space of *local* Wick polynomials $A(x)$. The TOP T_n are (multi)linear, totally symmetric maps from $W^{\otimes n}$ into the space of operator-valued tempered distributions,

$$T_n(A_1(x_1) \cdots A_n(x_n)) \in \mathcal{S}'(\mathbb{R}^{4n}),$$

⁵Incidentally, if Lorentz covariance is assumed on the level of *operators* (that is, for the fields and their TOP) and not only on the level of vacuum expectation values, local commutativity can be derived very easily. For the benefit of the reader this is done in an appendix.

satisfying the Bogoliubov–Shirkov–Epstein–Glaser axioms [17, 25, 27]. Besides Lorentz invariance, there is mainly causality:

$$T_n(A_1(x_1) \cdots A_n(x_n)) = T_l(A_1(x_1) \cdots A_l(x_l)) T_{n-l}(A_{l+1}(x_{l+1}) \cdots A_n(x_n)), \quad (3)$$

if $\{x_1, \dots, x_l\} \cap (\{x_{l+1}, \dots, x_n\} + \overline{V}_-) = \emptyset$. We use generating functionals of the form

$$\mathbf{T}(e^{A(h)}) := \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n T_n(A(x_1) \cdots A(x_n)) h(x_1) \cdots h(x_n); \quad (4)$$

in particular the \mathbf{S} -matrix:

$$\mathbf{S}(g) := \mathbf{T}(e^{iV(g)}), \quad g \in \mathcal{S}(\mathbb{R}^4), \quad (5)$$

for $V \in W$ a suitable first-order interaction Lagrangian.

Coupling constants have been replaced by Schwartz “switching functions” collectively denoted by g ; this aims to excise the infrared problem while dealing with the ultraviolet one. The \mathbf{S} -matrix operator on the Fock space of the incoming fields as a functional of g is a centrepiece of the theory. It acts as a generating function for the interacting fields by the Stepanov–Polivanov–Bogoliubov formula [25]. For the retarded ones:

$$A_{V(g)}(x) \equiv A_{\text{ret}}(g; x) := -i \frac{\delta}{\delta h(x)} \Big|_{h=0} \mathbf{S}(g)^{-1} \mathbf{T}(e^{i(V(g)+A(h))}) \quad \text{for } A \in W. \quad (6)$$

Both notations will be used. TOPs of these (retarded) interacting fields may as well be defined in an analogous way by higher derivatives with respect to h —see formulas (75) and (76) in [17]. They are all functionals of g . Retarded interacting fields are causal in the sense that

$$A_{V(g)+V_1(g_1)}(x) = A_{V(g)}(x) \quad \text{if } \text{supp } g_1 \cap (x + \overline{V}_-) = \emptyset; \quad (7)$$

where $V_1 \in W$ is arbitrary. With an obvious (re)normalization of the TOP, the interacting fields in causal perturbation theory satisfy the Yang–Feldman–Källén equations.⁶

From the causally renormalized interacting fields the interacting W -functions are still some way off. One has to perform the adiabatic limit as $g \uparrow 1$, bristling with difficulties [17, 30, 31].⁷

The good news is that local commutativity of the interacting fields can be proved rather straightforwardly prior to the adiabatic limit. It follows from the GLZ relation [14],

$$i[A_{V(g)}(x), B_{V(g)}(y)] = \frac{\delta}{\delta h(y)} \Big|_{h=0} A_{V(g)+B(h)}(x) - \frac{\delta}{\delta h(x)} \Big|_{h=0} B_{V(g)+A(h)}(y) \quad (A, B \in W), \quad (8)$$

⁶In the exact context of a well behaved background field this was verified in [28]. A full treatment of interacting fields for QED in causal perturbation theory was given in [29].

⁷We mention that obstructions to the adiabatic limit are discussed in a recent paper, invoking Fedosov’s index to relate causal renormalization and Wightman theory via Haag’s local algebraic formalism [32].

and the causality of the retarded interacting fields (7). In the present procedure the GLZ relation is a consequence of the definition (6) of the interacting fields —see Proposition 2 in [33]. Alternatively it can be taken as a defining axiom for perturbative interacting fields [15, 33]. Another piece of good news is that, provided that the infrared behaviour is good, one can work with the incoming Fock vacuum. In [17, 30] Epstein and Glaser were able to show that Wightman-like functions exist for purely massive, asymptotically complete models.

Now the bad news. There's the rub: as far as we know, the non-linear Wightman conditions have not been proved to hold in the Epstein–Glaser formalism (where fields and the state space itself are constructed as formal power series). It looks like a fearsome task, and it may well happen that imposing too good a behaviour leads back to overly strong restrictions on the nature of interaction. The matter of applicability of Jost's line of proof for PCT invariance in the causal framework is undecided as yet.

4 PCT invariance survives renormalization

Since the underlying issue is renormalization, turning from *blitzkrieg* to humble trench warfare, we expound a pertinent result.

Assumption 1. *The free fields are PCT-covariant, that is, there exists an anti-unitary operator Θ in the Fock space of free fields such that*

$$\Theta\Omega = \Omega \quad \text{and} \quad \Theta\Phi(x)\Theta^{-1} = \Phi^c(-x),$$

where Ω is the Fock vacuum and Φ^c is a suitable conjugate of the field Φ (passing to the adjoint and multiplication by a suitable matrix). Say, for a charged scalar field ϕ or for a Dirac field ψ :

$$\Theta\phi(x)\Theta^{-1} = \phi^\dagger(-x), \quad \Theta\psi(x)\Theta^{-1} = -i\gamma_5\psi^{\dagger T}(-x).$$

The PCT transformation is not usually an involution; for fermions it holds $\Theta^2 = (-1)^F$, where F is the number operator of the fields, which implies $\Theta^2\psi(x)\Theta^{-2} = -\psi(x)$ and $\Theta^4 = \mathbf{1}$. We will simply assume that

$$\Theta^{2N} = \mathbf{1} \quad \text{for some } N \in \mathbb{N} \setminus \{0\}. \quad (9)$$

Proposition 1. *For hermitian or charged scalar fields and for Dirac fields, normal ordering commutes with the PCT transformation, that is normally ordered products of free fields are also PCT-covariant:*

$$\Theta:\Phi(x_1)\cdots\Phi(x_n):\Theta^{-1} = :\Theta\Phi(x_1)\cdots\Phi(x_n)\Theta^{-1}: = :\Phi^c(-x_1)\cdots\Phi^c(-x_n):. \quad (10)$$

Proof. Proceeding by induction on n , we use Wick's theorem in the form

$$\begin{aligned} :\Phi(x_1)\cdots\Phi(x_n): &= :\Phi(x_1)\cdots\Phi(x_{n-1}): \Phi(x_n) \\ &\quad - \sum_{l=1}^{n-1} (\Omega, \Phi(x_l)\Phi(x_n)\Omega) :\Phi(x_1)\cdots\widehat{\Phi(x_l)}\cdots\Phi(x_{n-1}):. \end{aligned} \quad (11)$$

The notation means that $\Phi(x_l)$ is omitted. Since Θ is anti-unitary, the induction assumption yields

$$\begin{aligned} \Theta : \Phi(x_1) \cdots \Phi(x_n) : \Theta^{-1} &= : \Phi^c(-x_1) \cdots \Phi^c(-x_{n-1}) : \Phi^c(-x_n) \\ &- \sum_{l=1}^{n-1} (\Omega, \Phi(x_l) \Phi(x_n) \Omega)^* : \Phi^c(-x_1) \cdots \widehat{\Phi^c(-x_l)} \cdots \Phi^c(-x_{n-1}) : = : \Phi^c(-x_1) \cdots \Phi^c(-x_n) :, \end{aligned}$$

where we use that

$$(\Omega, \Phi(x_l) \Phi(x_n) \Omega)^* = (\Theta \Omega, \Theta \Phi(x_l) \Theta^{-1} \Theta \Phi(x_n) \Theta^{-1} \Theta \Omega) = (\Omega, \Phi^c(-x_l) \Phi^c(-x_n) \Omega)$$

and that (11) holds also for the field $\Phi^c(-x)$. \square

In this section we only study interactions V which are local Wick polynomials and scalar with respect to Lorentz transformations. One additionally needs that V be real: $V^\dagger(x) = V(x)$ on a dense subspace. In various cases the above proposition implies that V is PCT-invariant

$$\Theta V(x) \Theta^{-1} = V(-x). \quad (12)$$

If V is built from different kinds of free fields, it is perhaps not entirely clear that the mentioned conditions on V suffice for (12) to hold. Therefore, we take it as an assumption.

Assumption 2. *The interaction V is a local Wick polynomial $V \in W$, which is PCT-invariant: $\Theta V(x) \Theta^{-1} = V(-x)$.*

We turn to the PCT-transformation of TOP. Since PCT contains time reversal, we need to introduce the *antichronological products* $(\overline{T}_n)_{n \in \mathbb{N}}$. A sequence $(T_n)_{n \in \mathbb{N}}$ of TOP determines a pertinent sequence $(\overline{T}_n)_{n \in \mathbb{N}}$; the \overline{T}_n are also multilinear and totally symmetric maps defined by

$$\overline{\mathbf{T}}(e^{-A(h)}) := \mathbf{T}(e^{A(h)})^{-1}, \quad (13)$$

with $h \in \mathcal{S}(\mathbb{R}^4)$. In keeping with the previous notation,

$$\overline{\mathbf{T}}(e^{-A(h)}) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int dx_1 \cdots dx_n \overline{T}_n(A(x_1) \cdots A(x_n)) h(x_1) \cdots h(x_n).$$

As the term “antichronological” indicates, the \overline{T}_n satisfy (3), with the \overline{T} -products on the right hand side in *reverse order*.

Theorem 2. (a) *Let $A^c(-x) := \Theta A(x) \Theta^{-1}$. The time ordered products T_n can be (re)normalized in such a way that*

$$\Theta T_n(A_1(x_1) \cdots A_n(x_n)) \Theta^{-1} = \overline{T}_n(A_1^c(-x_1) \cdots A_n^c(-x_n)), \quad (14)$$

$$\Theta \overline{T}_n(A_1(x_1) \cdots A_n(x_n)) \Theta^{-1} = T_n(A_1^c(-x_1) \cdots A_n^c(-x_n)), \quad (15)$$

for arbitrary $A_i \in W$. That is to say, conjugation of T_n and \bar{T}_n by the PCT operator amounts to mutual exchange and conjugation of the arguments.

(b) The \mathbf{S} -matrix is PCT covariant:

$$\Theta \mathbf{S}(g) \Theta^{-1} = \mathbf{S}(\hat{g})^{-1} \quad \text{where} \quad \hat{g}(x) := g(-x). \quad (16)$$

(c) Advanced interacting fields

$$A_{\text{adv}}(g; x) := -i \frac{\delta}{\delta h(x)} \Big|_{h=0} \mathbf{T}(e^{i(V(g)+A(h))}) \mathbf{S}(g)^{-1}, \quad (17)$$

are mapped by the PCT transformation into retarded interacting fields (6) and viceversa:

$$\begin{aligned} \Theta A_{\text{adv}}(g; x) \Theta^{-1} &= A_{\text{ret}}^c(\hat{g}; -x); \\ \Theta A_{\text{ret}}(g; x) \Theta^{-1} &= A_{\text{adv}}^c(\hat{g}; -x). \end{aligned} \quad (18)$$

Remark 3. Advanced interacting fields are “anti-causal” in the sense that

$$A_{\text{adv}}(g + g_1; x) = A_{\text{adv}}(g; x) \quad \text{if} \quad \text{supp } g_1 \cap (x + \bar{V}_+) = \emptyset. \quad (19)$$

The support properties (7) and (19) of the retarded, respectively advanced interacting fields are consistent with (18).

Proof. a) \implies (b): This is obtained straightforwardly by using definitions (4), (5), (13), anti-linearity of Θ and (12).

(a) and (b) \implies (c): Due to

$$\frac{\delta}{\delta h(x)} \Big|_{h=0} \mathbf{T}(e^{i(V(g)+A(h))}) \bar{\mathbf{T}}(e^{-i(V(g)+A(h))}) = 0,$$

the advanced field (17) may alternatively be written as

$$A_{\text{adv}}(g; x) = i \frac{\delta}{\delta h(x)} \Big|_{h=0} \mathbf{S}(g) \bar{\mathbf{T}}(e^{-i(V(g)+A(h))}).$$

With that and with (a), (b) and anti-linearity of Θ we obtain

$$\Theta A_{\text{ret}}(g; x) \Theta^{-1} = i \frac{\delta}{\delta h(x)} \Big|_{h=0} \mathbf{S}(\hat{g}) \bar{\mathbf{T}}(e^{-i(V(\hat{g})+A^c(\hat{h}))}) = A_{\text{adv}}^c(\hat{g}; -x).$$

The second relation in (18) is proved analogously.

The proof of (a) goes by induction on n , following the Epstein–Glaser construction. In contrast to the latter, we do not use the distribution-splitting method; instead we borrow Stora’s extension of distributions method —see [23, 34, 35]. This shortens the discussion.

The case $n = 1$ follows from $T_1(A(x)) := A(x) =: \bar{T}_1(A(x))$.

Going to the inductive step $n - 1 \mapsto n$, the causality condition (3) determines

$$T_n^\circ(A_1(x_1) \cdots A_n(x_n)) := T_n(A_1(x_1) \cdots A_n(x_n)) \Big|_{S(\mathbb{R}^{4n} \setminus \Delta_n)}, \quad (20)$$

where $\Delta_n := \{(x_1, \dots, x_n) \in \mathbb{R}^{4n} \mid x_1 = \dots = x_n\}$, uniquely in terms of the lower orders $(T_{l < n})$. Renormalization of subgraphs is taken up by the inductive procedure. Since the $(T_{l < n})$ are PCT-covariant (14) by assumption, we obtain

$$\begin{aligned} \Theta T_n^\circ(A_1(x_1) \cdots A_n(x_n)) \Theta^{-1} &= \Theta T_l(A_1(x_1) \cdots A_l(x_l)) T_{n-l}(A_1(x_{l+1}) \cdots A_n(x_n)) \Theta^{-1} \\ &= \bar{T}_l(A_1^c(-x_1) \cdots A_l^c(-x_l)) \bar{T}_{n-l}(A_{l+1}^c(-x_{l+1}) \cdots A_n^c(-x_n)) = \bar{T}_n^\circ(A_1^c(-x_1) \cdots A_n^c(-x_n)), \end{aligned} \quad (21)$$

if $\{x_1, \dots, x_l\} \cap (\{x_{l+1}, \dots, x_n\} + \bar{V}_-) = \emptyset$, where \bar{T}° is defined like in (20) and in the last equality it is used that \bar{T}° factorizes antichronologically. In the same way one derives (15) for $\Theta \bar{T}_n^\circ \Theta^{-1}$.

It follows that PCT invariance (14) and (15) can be violated only in the extension

$$S'(\mathbb{R}^{4n} \setminus \Delta_n) \ni T_n^\circ(A_1(x_1) \cdots) \rightarrow T_n(A_1(x_1) \cdots) \in S'(\mathbb{R}^{4n}), \quad (22)$$

that is, in the process of renormalization. To obtain a PCT-covariant renormalization we take an arbitrary extension $T_n(A_1(x_1) \cdots) \in S'(\mathbb{R}^{4n})$ of $T_n^\circ(A_1(x_1) \cdots)$ fulfilling all other renormalization conditions (e.g. Poincaré covariance, unitarity, power counting) and symmetrize it with respect to the *finite* group generated by Θ [27, App. D]:

$$\begin{aligned} T_n^{\text{sym}}(A_1(x_1) \cdots A_n(x_n)) &:= \frac{1}{2N} \sum_{l=0}^{N-1} \left(\Theta^{2l} T_n(\Theta^{-2l} A_1(x_1) \Theta^{2l} \cdots \Theta^{-2l} A_n(x_n) \Theta^{2l}) \Theta^{-2l} \right. \\ &\quad \left. + \Theta^{2l+1} \bar{T}_n(\Theta^{-2l-1} A_1(x_1) \Theta^{2l+1} \cdots \Theta^{-2l-1} A_n(x_n) \Theta^{2l+1}) \Theta^{-(2l+1)} \right). \end{aligned} \quad (23)$$

This is also an extension of $T_n^\circ(A_1(x_1) \cdots)$, since $\Theta^{2l} T_n(\cdots) \Theta^{-2l}$ and $\Theta^{2l+1} \bar{T}_n(\cdots) \Theta^{-(2l+1)}$ are respectively extensions of $\Theta^{2l} T_n^\circ(\cdots) \Theta^{-2l}$ and $\Theta^{2l+1} \bar{T}_n^\circ(\cdots) \Theta^{-(2l+1)}$, and since

$$\begin{aligned} \Theta^{2l} T_n^\circ(\Theta^{-2l} A_1(x_1) \Theta^{2l} \cdots) \Theta^{-2l} &= T_n^\circ(A_1(x_1) \cdots), \quad \text{as well as} \\ \Theta^{2l+1} \bar{T}_n^\circ(\Theta^{-2l-1} A_1(x_1) \Theta^{2l+1} \cdots) \Theta^{-(2l+1)} &= T_n^\circ(A_1(x_1) \cdots), \end{aligned}$$

due to (21).

The only tricky part remaining is to show that the antichronological product \bar{T}_n^{sym} corresponding to T_n^{sym} according to its definition (13) can be written similarly as

$$\begin{aligned} \bar{T}_n^{\text{sym}}(A_1(x_1) \cdots A_n(x_n)) &= \frac{1}{2N} \sum_{l=0}^{N-1} \left(\Theta^{2l} \bar{T}_n(\Theta^{-2l} A_1(x_1) \Theta^{2l} \cdots \Theta^{-2l} A_n(x_n) \Theta^{2l}) \Theta^{-2l} \right. \\ &\quad \left. + \Theta^{2l+1} T_n(\Theta^{-2l-1} A_1(x_1) \Theta^{2l+1} \cdots \Theta^{-2l-1} A_n(x_n) \Theta^{2l+1}) \Theta^{-(2l+1)} \right). \end{aligned} \quad (24)$$

Indeed, using this and (9), one sees that the T_n^{sym} and the corresponding \bar{T}_n^{sym} fulfil the assertions (14) and (15).

To prove (24) we use that any extension T_n and the corresponding \bar{T}_n (13) satisfy

$$\begin{aligned} [T_n + (-1)^n \bar{T}_n] (A_i(x_i)_{i \in \bar{n}}) &= \sum_{M \subset \bar{n}, 1 \leq |M| \leq n-1} (-1)^{|M|+1} \bar{T}_{|M|} (A_i(x_i)_{i \in M}) T_{n-|M|} (A_j(x_j)_{j \in \bar{n} \setminus M}) \\ &= \sum_{M \subset \bar{n}, 1 \leq |M| \leq n-1} (-1)^{n-|M|+1} T_{|M|} (A_i(x_i)_{i \in M}) \bar{T}_{n-|M|} (A_j(x_j)_{j \in \bar{n} \setminus M}), \end{aligned} \quad (25)$$

where $\bar{n} := \{1, \dots, n\}$ and $|M|$ is the size of block M ; these are just the relations $\bar{\mathbf{T}}(e^{-A(h)}) \mathbf{T}(e^{A(h)}) = \mathbf{1} = \mathbf{T}(e^{A(h)}) \bar{\mathbf{T}}(e^{-A(h)})$. Since the two expressions on the right hand side of (25) are inductively given, the proof is complete if we succeed to show that expression (24) for \bar{T}_n^{sym} fulfils

$$T_n^{\text{sym}} + (-1)^n \bar{T}_n^{\text{sym}} = T_n + (-1)^n \bar{T}_n, \quad (26)$$

for the arbitrary extension T_n used in (23). From (25) and PCT invariance of the lower orders ($T_{l < n}$) and ($\bar{T}_{l < n}$) we obtain

$$\begin{aligned} &\Theta [\bar{T}_n + (-1)^n T_n] (A_i(x_i)_{i \in \bar{n}}) \Theta^{-1} \\ &= \sum_{M \subset \bar{n}, 1 \leq |M| \leq n-1} (-1)^{n-|M|+1} T_{|M|} (\Theta A_i(x_i)_{i \in M} \Theta^{-1}) \bar{T}_{n-|M|} (\Theta A_j(x_j)_{j \in \bar{n} \setminus M} \Theta^{-1}) \\ &= [T_n + (-1)^n \bar{T}_n] (\Theta A_i(x_i)_{i \in \bar{n}} \Theta^{-1}). \end{aligned}$$

As well,

$$\Theta [T_n + (-1)^n \bar{T}_n] (A_i(x_i)_{i \in \bar{n}}) \Theta^{-1} = [\bar{T}_n + (-1)^n T_n] (\Theta A_i(x_i)_{i \in \bar{n}} \Theta^{-1}).$$

Using these relations in the sum of (23) and (24), we indeed get (26):

$$\begin{aligned} [T_n^{\text{sym}} + (-1)^n \bar{T}_n^{\text{sym}}] (A_1(x_1) \cdots) &= \frac{1}{2N} \sum_{l=0}^{N-1} \left(\Theta^{2l} [T_n + (-1)^n \bar{T}_n] (\Theta^{-2l} A_1(x_1) \Theta^{2l} \cdots) \Theta^{-2l} \right. \\ &\quad \left. + \Theta^{2l+1} [\bar{T}_n + (-1)^n T_n] (\Theta^{-2l-1} A_1(x_1) \Theta^{2l+1} \cdots) \Theta^{-(2l+1)} \right) \\ &= [T_n + (-1)^n \bar{T}_n] (A_1(x_1) \cdots). \end{aligned} \quad \square$$

5 Conclusion

The argument in reference [5] fails to grapple with the nitty-gritty of renormalization: it presumes that suitably renormalized interacting fields exist such that Wightman's axioms 0 to II are fulfilled. Instead a reasonable avenue is to concentrate just on proving or disproving PCT invariance of the TOP for large enough classes of models, constructed as rigorously as possible in perturbation theory by dealing with renormalization through the causal method.

We have precisely shown how PCT propagates through causal perturbative renormalization. It pertains to point out caveats for our theorem. To begin with, if someone ever were clever enough to come out with a non-invariant T_1 , there is nothing to do. We have focused on the TOP. For *physical*

PCT conservation to ensue, the adiabatic limit of the model after renormalization must exist. As pointed out by Kobayashi and Sanda time ago [36], the remit of any approach to PCT conservation is narrowed by the fact that both heuristic and rigorous proofs make use of properties of asymptotic states of particles that just do not apply in QCD, whereupon the elementary excitations of the field are confined. One has to admit that large parts of the Standard Model disown the basic hypothesis of any **S**-matrix theory.

Even for a garden-variety model like QED, the adiabatic limit [31] exists only in a weak sense. In addition, as conclusively shown by Herdegen, Dirac fields respecting Gauss' law which are only "spatially local" provide the best tool to construct QED, free from the infrared catastrophe [37]. We believe that this should not decisively impinge on the issues considered here [38]; but the question is open to debate. It should be stressed that dropping locality of the interactions, PCT non-conservation and Lorentz invariance can perfectly coexist [39, 40]. Now, the borderline between local and non-local models is not nearly as neat as one would like. It would appear that local and non-local fields may share the same **S**-matrix [41], for that matter.

Appendix. From Lorentz covariance to local commutativity

To simplify the notation we consider a (possibly interacting) scalar field $\phi(x)$.

Assumption 3. $\phi(x)$ is L_+^\uparrow -covariant, that is, there exists a representation $L_+^\uparrow \ni \Lambda \mapsto U(\Lambda)$ such that

$$\phi(\Lambda x) = U(\Lambda) \phi(x) U(\Lambda)^{-1} \quad \text{for all } \Lambda \in L_+^\uparrow.$$

In addition the TOP

$$T(\phi(x) \phi(y)) := H(x^0 - y^0) \phi(x) \phi(y) + H(y^0 - x^0) \phi(y) \phi(x) \quad \text{for all } x^0 \neq y^0$$

(where H denotes the Heaviside function) is L_+^\uparrow -covariant

$$T(\phi(\Lambda x) \phi(\Lambda y)) = U(\Lambda) T(\phi(x) \phi(y)) U(\Lambda)^{-1} \quad \text{for all } x \neq y. \quad (27)$$

In particular, assumption (27) means that for $x^0 = y^0$ and $x \neq y$ the definition

$$T(\phi(x) \phi(y)) := U(\Lambda)^{-1} T(\phi(\Lambda x) \phi(\Lambda y)) U(\Lambda)$$

is independent of the choice of $\Lambda \in L_+^\uparrow$ with $(\Lambda x)^0 \neq (\Lambda y)^0$.

The following derivation of *local commutativity* (2) is motivated by [42]. Let x, y be given such that $(x - y)^2 < 0$. We choose $\Lambda_1, \Lambda_2 \in L_+^\uparrow$ such that $(\Lambda_1 x)^0 > (\Lambda_1 y)^0$ and $(\Lambda_2 x)^0 < (\Lambda_2 y)^0$. With that one obtains

$$\begin{aligned} \phi(x) \phi(y) &= U(\Lambda_1)^{-1} \phi(\Lambda_1 x) \phi(\Lambda_1 y) U(\Lambda_1) = U(\Lambda_1)^{-1} T(\phi(\Lambda_1 x) \phi(\Lambda_1 y)) U(\Lambda_1) \\ &= T(\phi(x) \phi(y)) = U(\Lambda_2)^{-1} T(\phi(\Lambda_2 x) \phi(\Lambda_2 y)) U(\Lambda_2) \\ &= U(\Lambda_2)^{-1} \phi(\Lambda_2 y) \phi(\Lambda_2 x) U(\Lambda_2) = \phi(y) \phi(x). \quad \square \end{aligned}$$

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⁸But widely known.